Part IV - Lognormally-Distributed Portfolio Value

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If asset returns are normally-distributed then asset values are lognormally-distributed. The value of a basket of assets (a portfolio) equals the sum of the values of the individual assets in that basket. A problem arises in that the sum of lognormally-distributed asset values does not follow a known distribution. Whereas asset values are lognormally-distributed the value of the basket is not. If the random variable $A_i(t)$ is the value of the i'th asset in the portfolio at time t, the random variable $P(t)$ is the value of the portfolio at time t, and the variable N is the number of assets in the portfolio, then our problem in equation form is...

$$A_i(t) = A_i(0)\,\mathrm{Exp}\left\{\text{return}\right\} \sim \text{Lognormal Distribution} \;\; \text{...but...} \;\; P(t) = \sum_{i=1}^{N} A_i(t) \sim \text{Unknown Distribution} \tag{1}$$

Approximating the sum of two or more lognormal random variables by a single lognormal random variable is useful in practice as that approximation can be used to model securities involving a basket of assets. For example, the payoff on a basket option depends on the total value of the assets in the basket. We can value a basket option by approximating the total value of the assets in the basket via a single lognormal variable and then use the Black-Scholes model to value the option. We can also construct a VaR (value at risk) analysis using the same methodology.

We will define the variable λ to be the portfolio's return mean, the variable σ to be the portfolio's return volatility, and the variable t to be time in years. In this white paper we will develop the mathematics to approximate the sum of two or more lognormal random variables by a single lognormal random variable using The Moment Matching Method. To that end we will use the following hypothetical problem from Part II...

Our Hypothetical Problem

We are tasked with building an portfolio value model that assumes that asset prices are normally-distributed. We are given the following go-forward model assumptions...

Table 1: Portfolio Composition

Description	Asset 1	Asset 2	Asset 3	Total
Asset value at time zero (in dollars)	300,000	500,000	200,000	1,000,000
Annual return mean	12.00%	10.00%	8.00%	–
Annual return volatility	30.00%	20.00%	10.00%	–
Annual distribution rate	5.00%	4.00%	3.00%	–
Common factor correlations	0.6928	0.8660	0.5774	–

Using the parameters in Table 1 above our model parameters are...

Table 2: Continuous-Time Model Parameters

Symbol	Asset 1	Asset 2	Asset 3	Total	Notes
$A_i(0)$	300	500	200	1,000	Dollars in thousands
λ_i	0.0227	0.0383	0.0438	–	Continuous-time return mean (see Equation (2))
σ_i	0.3000	0.2000	0.1000	–	Continuous-time return volatility
ρ_i	0.6928	0.8660	0.5774	–	Correlation with common factor
w_i	0.3000	0.5000	0.2000	1.0000	Portfolio weights

Note the following discrete-time to continuous-time conversion...

$$\lambda_i = \ln\left(1 + \text{discrete-time annual return mean - discrete-time annual distribution rate}\right) - \frac{1}{2}\,\sigma^2 \tag{2}$$

Our task is to answer the following questions:

Question 1: What is portfolio value mean and variance at the end of year three?

Question 2: What is portfolio return mean and variance at the end of year three?

Question 3: What is the random portfolio value equation?

Individual Asset Value

We defined the variable $A_i(t)$ to be the value of the i'th asset in the portfolio at time t. We will define the variable λ_i to be the i'th asset's continuous-time return mean minus the distribution rate, the variable σ_i to be the i'th asset's return volatility, and the variable $\delta W_i(t)$ to be the change in the driving brownian motion at time t. The stochastic differential equation (SDE) that defines the change in asset value excluding correlation is...

$$\delta A_i(t) = \lambda_i\, A_i(t)\,\delta t + \sigma_i\, A_i(t)\,\delta W_i(t) \;\text{ ...where... }\; \delta W_i(t) \sim N\left[0, \delta t\right] \tag{3}$$

We will model correlation by adding the variable $\delta W_c(t)$, which is the change in a brownian motion that is common to all assets in the portfolio (i.e. a common factor), and the variable ρ_i, which is the correlation between the i'th asset's return and the common factor, to the equation above. Using Equation (3) above the revised equation for the change in asset value including correlation is... [1]

$$\delta A_i(t) = \lambda_i\, A_i(t)\,\delta t + \rho_i\,\sigma_i\, A_i(t)\,\delta W_c(t) + \sqrt{1-\rho_i^2}\,\sigma_i\, A_i(t)\,\delta W_i(t) \;\text{ ...where... }\; \delta W_c(t) \sim N\left[0, \delta t\right] \tag{4}$$

We can standardize Equation (4) above by rewriting it as...

$$\delta A_i(t) = \lambda_i\, A_i(t)\,\delta t + \rho_i\,\sigma_i\sqrt{\delta t}\, A_i(t)\, Z_c + \sqrt{1-\rho_i^2}\,\sigma_i\sqrt{\delta t}\, A_i(t)\, Z_i \;\text{ ...where... }\; Z_i \sim N\left[0, 1\right] \text{ and } Z_c \sim N\left[0, 1\right] \tag{5}$$

Note that in Equation (5) above we have the following expectations...

$$\mathbb{E}\left[Z_i\right] = \mathbb{E}\left[Z_c\right] = 0 \;\text{ ...and... }\; \mathbb{E}\left[Z_i^2\right] = \mathbb{E}\left[Z_c^2\right] = 1 \;\text{ ...and... }\; \mathbb{E}\left[Z_c\, Z_i\right] = 0 \tag{6}$$

Given that Equation (5) above is the differential equation that defines the change in random asset value at time t then the solution to that differential equation is the equation for random asset value at time t, which is...

$$A_i(t) = A_i(0)\,\mathrm{Exp}\left\{\lambda_i\, t + \rho_i\,\sigma_i\sqrt{t}\, Z_c + \sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\, Z_i\right\} \tag{7}$$

If we divide both sides of Equation (7) above by the known asset value at time zero then we get the following equation...

$$\frac{A_i(t)}{A_i(0)} = \mathrm{Exp}\left\{\lambda_i\, t + \rho_i\,\sigma_i\sqrt{t}\, Z_c + \sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\, Z_i\right\} \tag{8}$$

Using Appendix Equation (67) below the equation for the first moment of the distribution of random asset value at time t divided by the known asset value at time zero is...

$$\mathbb{E}\left[\frac{A_i(t)}{A_i(0)}\right] = \mathrm{Exp}\left\{\left(\lambda_i + \frac{1}{2}\,\sigma_i^2\right)t\right\} \tag{9}$$

Using Equation (7) above the equation for the square of random asset value at time t is...

$$A_i(t)^2 = A_i(0)^2\,\mathrm{Exp}\left\{2\left(\lambda_i\, t + \rho_i\,\sigma_i\sqrt{t}\, Z_c + \sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\, Z_i\right)\right\} \tag{10}$$

If we divide both sides of Equation (10) above by the square of the known asset value at time zero then we get the following equation...

$$\frac{A_i(t)^2}{A_i(0)^2} = \text{Exp}\left\{2\left(\lambda_i\, t + \rho_i\, \sigma_i \sqrt{t}\, Z_c + \sqrt{1-\rho_i^2}\, \sigma_i \sqrt{t}\, Z_i\right)\right\} \tag{11}$$

Using Appendix Equation (82) below the equation for the second moment of the distribution of random asset value at time t divided by the known asset value at time zero is...

$$\mathbb{E}\left[\frac{A_i(t)^2}{A_i(0)^2}\right] = \text{Exp}\left\{2\left(\lambda_i\, t + \sigma_i^2\, t\right)\right\} \tag{12}$$

Using Equation (7) above the equation for the product of the i'th and j'th asset values at time t where $i \neq j$ is...

$$\begin{aligned} A_i(t)A_j(t) &= A_i(0)\,\text{Exp}\left\{\lambda_i\, t + \rho_i\, \sigma_i \sqrt{t}\, Z_c + \sqrt{1-\rho_i^2}\, \sigma_i \sqrt{t}\, Z_i\right\} A_j(0)\,\text{Exp}\left\{\lambda_j\, t + \rho_j\, \sigma_j \sqrt{t}\, Z_c + \sqrt{1-\rho_j^2}\, \sigma_j \sqrt{t}\, Z_j\right\} \\ &= A_i(0)A_j(0)\,\text{Exp}\left\{\left(\lambda_i + \lambda_j\right) t + \left(\rho_i\, \sigma_i \sqrt{t} + \rho_j\, \sigma_j \sqrt{t}\right) Z_c + \sqrt{1-\rho_i^2}\, \sigma_i \sqrt{t}\, Z_i + \sqrt{1-\rho_j^2}\, \sigma_j \sqrt{t}\, Z_j\right\} \end{aligned} \tag{13}$$

If we divide both sides of Equation (13) above by the product of the known asset values at time zero of the i'th and j'th asset then we get the following equation...

$$\frac{A_i(t)A_j(t)}{A_i(0)A_j(0)} = \text{Exp}\left\{\left(\lambda_i + \lambda_j\right) t + \left(\rho_i\, \sigma_i \sqrt{t} + \rho_j\, \sigma_j \sqrt{t}\right) Z_c + \sqrt{1-\rho_i^2}\, \sigma_i \sqrt{t}\, Z_i + \sqrt{1-\rho_j^2}\, \sigma_j \sqrt{t}\, Z_j\right\} \tag{14}$$

Using Appendix Equation (96) below the equation for the expected product of any two random asset values at time t where $i \neq j$ is...

$$\mathbb{E}\left[\frac{A_i(t)A_j(t)}{A_i(0)A_j(0)}\right] = \text{Exp}\left\{\left(\lambda_i + \lambda_j\right) t + \frac{1}{2}\left(\rho_i\, \sigma_i \sqrt{t} + \rho_j\, \sigma_j \sqrt{t}\right)^2\right\} \text{Exp}\left\{\frac{1}{2}\,(1-\rho_i^2)\, \sigma_i^2\, t\right\} \text{Exp}\left\{\frac{1}{2}\,(1-\rho_j^2)\, \sigma_j^2\, t\right\} \tag{15}$$

Using Equations (9) and (12) above the equations for the first and second moments of the distribution of random asset value at time t are...

$$\mathbb{E}\left[A_i(t)\right] = A_i(0)\,\mathbb{E}\left[\frac{A_i(t)}{A_i(0)}\right] \text{ ...and... } \mathbb{E}\left[A_i(t)^2\right] = A_i(0)^2\,\text{Exp}\left\{2\left(\lambda_i\, t + \sigma_i^2\right) t\right\} \tag{16}$$

Individual Asset Return

We will define the variable $r_i(t)$ to be the random rate of return on asset A_i over the time interval $[0, t]$. The equation for individual asset return is...

$$\text{if... } A_i(t) = A_i(0)\,\text{Exp}\left\{r_i(t)\right\} \text{ ...then... } r_i(t) = \ln\left(A_i(t)\right) - \ln\left(A_i(0)\right) \tag{17}$$

Using Equation (7) above we can rewrite Equation (17) above as...

$$\begin{aligned} r_i(t) &= \ln\left(A_i(0)\,\text{Exp}\left\{\lambda_i\, t + \rho_i\, \sigma_i \sqrt{t}\, Z_c + \sqrt{1-\rho_i^2}\, \sigma_i \sqrt{t}\, Z_i\right\}\right) - \ln\left(A_i(0)\right) \\ &= \ln\left(A_i(0)\right) + \lambda_i\, t + \rho_i\, \sigma_i \sqrt{t}\, Z_c + \sqrt{1-\rho_i^2}\, \sigma_i \sqrt{t}\, Z_i - \ln\left(A_i(0)\right) \\ &= \lambda_i\, t + \rho_i\, \sigma_i \sqrt{t}\, Z_c + \sqrt{1-\rho_i^2}\, \sigma_i \sqrt{t}\, Z_i \end{aligned} \tag{18}$$

Using Appendix Equation (97) below the first moment of the distribution of the i'th asset's random rate of return is...

$$\mathbb{E}\left[r_i(t)\right] = \lambda_i\, t \tag{19}$$

Using Appendix Equation (98) below the second moment of the distribution of the i'th asset's random rate of return is...

$$\mathbb{E}\left[r_i(t)^2\right] = \lambda_i^2\, t^2 + \sigma_i^2\, t \tag{20}$$

Using Equation (19) above the mean of the distribution of the i'th asset's random rate of return is...

$$\text{mean of } r_i(t) = \mathbb{E}\bigg[r_i(t)\bigg] = \lambda_i\, t \tag{21}$$

Using Equations (19) and (20) above the variance of the distribution of the i'th asset's random rate of return is...

$$\text{variance of } r_i(t) = \mathbb{E}\bigg[r_i(t)^2\bigg] - \bigg[\mathbb{E}\bigg[r_i(t)\bigg]\bigg]^2 = \sigma_i^2\, t \tag{22}$$

Using Appendix Equation (99) below the expected product of the rates of return on the i'th and j'th asset over the time interval $[0, t]$ is...

$$\mathbb{E}\bigg[r_i(t)\, r_j(t)\bigg] = \lambda_i\, \lambda_j\, t^2 + \rho_i\, \rho_j\, \sigma_i\, \sigma_j\, t \tag{23}$$

Using Equations (19) and (23) above the covariance of the distribution of the product of the i'th and j'th asset's random rate of return is...

$$\text{covariance of } r_i(t) \text{ and } r_j(t) = \mathbb{E}\bigg[r_i(t)\, r_j(t)\bigg] - \mathbb{E}\bigg[r_i(t)\bigg]\,\mathbb{E}\bigg[r_j(t)\bigg] = \lambda_i\, \lambda_j\, t^2 + \rho_i\, \rho_j\, \sigma_i\, \sigma_j\, t - \lambda_i\, \lambda_j\, t^2 \tag{24}$$

Using Equation (24) above the equation for the pairwise correlation between the random rates of return on the i'th and j'th asset is...

$$\text{correlation of } r_i(t) \text{ and } r_j(t) = \frac{\text{covariance of } r_i(t) \text{ and } r_j(t)}{\text{volatility } r_i(t) \times \text{volatility } r_j(t)} = \frac{\lambda_i\, \lambda_j\, t^2 + \rho_i\, \rho_j\, \sigma_i\, \sigma_j\, t - \lambda_i\, \lambda_j\, t^2}{\sigma_i\, \sqrt{t}\, \sigma_j\, \sqrt{t}} = \rho_i\, \rho_j \tag{25}$$

Note that whereas the correlation between the rate of return on the i'th asset in the portfolio and the common factor is ρ_i, the correlation between the rates of return on any two assets in the portfolio is $\rho_i\, \rho_j$.

The Portfolio Value Equation

We defined the variable $P(t)$ to be the value of the portfolio of assets at time t. We will drop the subscripts and define the variable λ to be the portfolio's return mean minus the distribution rate, the variable σ to be the portfolio's return volatility, and the variable $\delta W(t)$ to be the change in the driving brownian motion at time t. The stochastic differential equation that defines the change in portfolio value at time t is...

$$\delta P(t) = \lambda\, P(t)\, \delta t + \sigma\, P(t)\, \delta W(t) \;\; \text{...where...} \;\; \delta W(t) \sim N\bigg[0, \delta t\bigg] \tag{26}$$

We can standardize Equation (26) above by rewriting it as...

$$\delta P(t) = \lambda\, P(t)\, \delta t + \sigma\, P(t)\, \sqrt{\delta t}\, Z \;\; \text{...where...} \;\; Z \sim N\bigg[0, 1\bigg] \tag{27}$$

The solution to Equation (27) above is the equation for random portfolio value at time t, which is...

$$P(t) = P(0)\, \text{Exp}\bigg\{\lambda\, t + \sigma \sqrt{t}\, Z\bigg\} \;\; \text{...where...} \;\; P(0) = \sum_{i=1}^{N} A_i(0) \;\; \text{...and...} \;\; N = \text{number of assets} \tag{28}$$

If portfolio return is normally-distributed then portfolio value is lognormally-distributed. Given the known properties of the lognormal distribution the first and second moments of the distribution of portfolio value as defined by Equation (28) above are... [2]

$$\mathbb{E}\bigg[P(t)\bigg] = P(0)\, \text{Exp}\bigg\{\bigg(\lambda + \frac{1}{2}\, \sigma^2\bigg) t\bigg\} \;\; \text{...and...} \;\; \mathbb{E}\bigg[P(t)^2\bigg] = P(0)^2\, \text{Exp}\bigg\{2\bigg(\lambda + \sigma^2\bigg) t\bigg\} \tag{29}$$

Since portfolio value at time t equals the sum of the values of the individual assets in the portfolio at time t then using Equation (7) above we can rewrite portfolio value Equation (28) above as...

$$P(t) = A_1(t) + A_2(t) + ... + A_N(t) = \sum_{i=1}^{N} A_i(t) \;\; \text{...where...} \;\; N = \text{number of assets} \tag{30}$$

Note that we can rewrite Equation (30) above as...

$$P(t) = A_1(0)\,\frac{A_1(t)}{A_1(0)} + A_2(0)\,\frac{A_2(t)}{A_2(0)} + ... + A_N(0)\,\frac{A_N(t)}{A_N(0)} = \sum_{i=1}^{N} A_i(0)\,\frac{A_i(t)}{A_i(0)} \tag{31}$$

We will rewrite Equation (31) above to be a function of portfolio value at time zero...

$$P(t) = P(0)\left[w_1\,\frac{A_1(t)}{A_1(0)} + w_2\,\frac{A_2(t)}{A_2(0)} + ... + w_N\,\frac{A_N(t)}{A_N(0)}\right] = P(0)\sum_{i=1}^{N} w_i\,\frac{A_i(t)}{A_i(0)} \text{ ...where... } w_i = \frac{A_i(0)}{P(0)} \tag{32}$$

We will define the variable $M_1(t)$ to be the first moment of the distribution of random portfolio value at time t. Using Equations (9) and (32) above the equation for the first moment is...

$$M_1(t) = \mathbb{E}\Big[P(t)\Big] = \mathbb{E}\left[P(0)\sum_{i=1}^{N} w_i\,\frac{A_i(t)}{A_i(0)}\right] = P(0)\sum_{i=1}^{N} w_i\,\mathbb{E}\left[\frac{A_i(t)}{A_i(0)}\right] \tag{33}$$

We will define the variable $\vec{\mathbf{w}}$ to be a vector of portfolio weights. The equation for this vector given that in our hypothetical problem the variable $N=3$ is...

$$\vec{\mathbf{w}} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \text{ ...where... } w_i = \frac{A_i(0)}{P(0)} \text{ ...and... } \sum_{i=1}^{N} w_i = 1 \tag{34}$$

We will define the variable $\vec{\mathbf{v}}$ to be a vector of first moments of random asset value at time t divided by the known asset value at time zero. The equation for this vector given that $N=3$ is...

$$\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \text{ ...where... } v_i = \mathbb{E}\left[\frac{A_i(t)}{A_i(0)}\right] \tag{35}$$

Using Equations (34) and (35) above we can rewrite Equation (33) above as...

$$M_1(t) = P(0)\,\vec{\mathbf{w}}^T\vec{\mathbf{v}} \tag{36}$$

Using the hypothetical problem parameters in Table 1 above and Equations (9) and (36) above the first moment of the distribution of random portfolio value at the end of year five for our Hypothetical Problem is...

$$\text{if... } \vec{\mathbf{w}} = \begin{bmatrix} 0.30 \\ 0.50 \\ 0.20 \end{bmatrix} \text{ ...and.. } \vec{\mathbf{v}} = \begin{bmatrix} 1.2250 \\ 1.1910 \\ 1.1576 \end{bmatrix} \text{ ...then... } M_1(5) = 1,000 \times \vec{\mathbf{w}}^T\vec{\mathbf{v}} = 1,195 \tag{37}$$

Using Equation (30) above the equation for the square of random portfolio value at time t is...

$$P(t)^2 = \Big(A_1(t) + A_2(t) + ... + A_N(t)\Big)^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} A_i(t)\,A_j(t) \tag{38}$$

Note that using Equation (32) above as our guide we can rewrite Equation (38) above as...

$$P(t)^2 = \left(P(0)\sum_{i=1}^{N} w_i\,\frac{A_i(t)}{A_i(0)}\right)^2 = P(0)^2\sum_{i=1}^{N}\sum_{j=1}^{N} w_i\,w_j\,\frac{A_i(t)}{A_i(0)}\,\frac{A_j(t)}{A_j(0)} \tag{39}$$

We will define the variable $M_2(t)$ to be the second moment of the distribution of random portfolio value at time t. Using Equation (39) above the equation for the second moment is...

$$M_2(t) = \mathbb{E}\Big[P(t)^2\Big] = \mathbb{E}\left[P(0)^2\sum_{i=1}^{N}\sum_{j=1}^{N} w_i\,w_j\,\frac{A_i(t)}{A_i(0)}\,\frac{A_j(t)}{A_j(0)}\right] = P(0)^2\sum_{i=1}^{N}\sum_{j=1}^{N} w_i\,w_j\,\mathbb{E}\left[\frac{A_i(t)}{A_i(0)}\,\frac{A_j(t)}{A_j(0)}\right] \tag{40}$$

We will define the variable $\mathbf{A}$ to be a matrix of expected products. Using Equations (12) and (15) above the equation for this matrix is...

$$\mathbf{A}=\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \text{ ...where... } \left[a_{i,j}\,\middle|\, i=j\right]=\mathbb{E}\left[\frac{A_i(t)^2}{A_i(0)^2}\right] \text{ ...and... } \left[a_{i,j}\,\middle|\, i\neq j\right]=\mathbb{E}\left[\frac{A_i(t)}{A_i(0)}\,\frac{A_j(t)}{A_j(0)}\right] \tag{41}$$

Using Equations (34) and (41) above we can rewrite Equation (40) above as...

$$M_2(t)=P(0)^2\,\vec{\mathbf{w}}^T\mathbf{A}\,\vec{\mathbf{w}} \tag{42}$$

Using the parameters in Table 1 above and Equations (12), (15) and (42) above the second moment of the distribution of random portfolio value at the end of year seven for our Hypothetical Problem is...

$$\text{if... } \mathbf{A}=\begin{bmatrix} 1.9659 & 1.6254 & 1.4701 \\ 1.6254 & 1.5994 & 1.4207 \\ 1.4701 & 1.4207 & 1.3809 \end{bmatrix} \text{ ...then... } M_2(5)=1,000^2\times\vec{\mathbf{w}}^T\mathbf{A}\,\vec{\mathbf{w}}=1,580,200 \tag{43}$$

Portfolio Value - The Moment Matching Method

We will estimate the values of λ and σ in Equation (28) above by matching the first two moments of random portfolio value (Equation (29) above) to the first two moments of the sum of individual asset values (Equations (36) and (42) above). This statement in equation form is...

$$P(0)\,\mathrm{Exp}\left\{\left(\lambda+\frac{1}{2}\,\sigma^2\right)t\right\}=M_1(t) \text{ ...and... } \bar{P}(0)^2\,\mathrm{Exp}\left\{2\left(\lambda+\sigma^2\right)t\right\}=M_2(t) \tag{44}$$

If we take the log of Equation (44) above then we have the following two simultaneous equations...

$$\ln(P(0))+\lambda\,t+\frac{1}{2}\,\sigma^2\,t=\ln(M_1(t)) \text{ ...and... } 2\,\ln(P(0))+2\,\lambda\,t+2\,\sigma^2\,t=\ln(M_2(t)) \tag{45}$$

Note that we can rewrite Equation (45) above as...

$$\lambda+\frac{1}{2}\,\sigma^2=\frac{\ln(M_1(t))-\ln(P(0))}{t} \text{ ...and... } 2\,\lambda+2\,\sigma^2=\frac{\ln(M_2(t))-2\,\ln(P(0))}{t} \tag{46}$$

We will make the following matrix and vector definitions...

$$\mathbf{Q}=\begin{bmatrix} 1.00 & 0.50 \\ 2.00 & 2.00 \end{bmatrix} \text{ ...and... } \vec{\mathbf{r}}=\begin{bmatrix} \lambda \\ \sigma^2 \end{bmatrix} \text{ ...and... } \vec{\mathbf{s}}=\begin{bmatrix} [\ln(M_1(t))-\ln(P(0))]\,/\,t \\ [\ln(M_2(t))-2\,\ln(P(0))]\,/\,t \end{bmatrix} \tag{47}$$

Using Equation (47) above we can rewrite Equation (46) above in the following matrix:vector notation...

$$\mathbf{Q}\,\vec{\mathbf{r}}=\vec{\mathbf{s}} \text{ ...such that... } \vec{\mathbf{r}}=\mathbf{Q}^{-1}\,\vec{\mathbf{s}} \text{ ...where... } \mathbf{Q}^{-1}=\begin{bmatrix} 2.00 & -0.50 \\ -2.00 & 1.00 \end{bmatrix} \tag{48}$$

Before we can use a single lognormally-distributed random variable to model portfolio value we need the portfolio's expected rate of return and return volatility. We do that by matching the moments of an unknown distribution to the moments of a known distribution. The values of the portfolio's return mean and return volatility are contained in vector $\vec{\mathbf{r}}$, which we solved for above. Using the parameters in Table 1 above and Equations (34), (**??**) and (**??**) above the value of vector $\vec{\mathbf{r}}$ for our Hypothetical Problem is...

$$\text{if... } \vec{\mathbf{s}}=\begin{bmatrix} [\ln(1,195)-\ln(1,000)]\,/\,3.00 \\ [\ln(1,580,200)-2\times\ln(1,000,000))]\,/\,3.00 \end{bmatrix}=\begin{bmatrix} 0.0593 \\ 0.1525 \end{bmatrix} \text{ ...then... } \vec{\mathbf{r}}=\mathbf{Q}^{-1}\,\vec{\mathbf{s}}=\begin{bmatrix} 0.0423 \\ 0.0340 \end{bmatrix} \tag{49}$$

Using Equation (49) above the estimated values of μ_p and σ_p are...

$$\mu_p=0.0423 \text{ ...and... } \sigma_p=\sqrt{0.0340}=0.1844 \tag{50}$$

The Answers To Our Hypothetical Problem

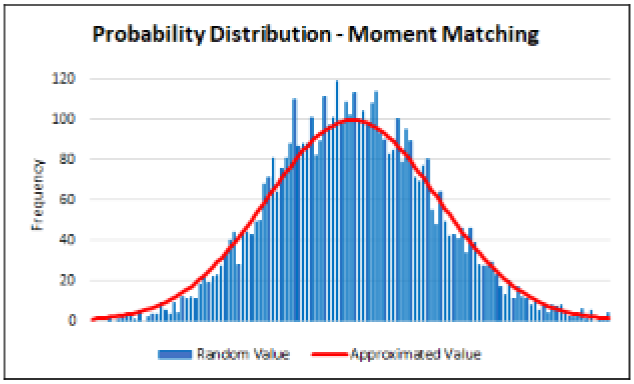
Question 1: What is expected portfolio value at the end of year 5?

Using the parameters in Table 1 above and Equation (50) above portfolio value at the end of year seven via the Moment Matching Method is...

$$\mathbb{E}\Big[P(t)\Big] = 1,000,000 \times \text{Exp}\left\{\left(0.0423 + \frac{1}{2} \times 0.1844^2\right) \times 3.00\right\} = 1,195,000 \tag{51}$$

Question 2: Show that The Moment Matching Method gives us a good approximation of portfolio value at the end of year 5.

Using Equations (7) and (30) to conduct 5,000 trials of random portfolio values (blue bars). Use the estimates for λ and σ in Equation (50) above to approximate random portfolio value (red line). The graph of the two methods is...



Note that The Moment Matching Method gives us a good approximation of the sample distribution of portfolio value.

References

[1] Gary Schurman, *The Gaussian Copula,* September, 2010

[2] Gary Schurman, *The Lognormal Distribution,* June, 2015

Appendix

A. The equation for the probability density function of a standardized normal distribution is...

$$f(z) = \sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\left\{-\frac{1}{2}\,z^2\right\} \text{ ...such that... } \int\limits_{-\infty}^{\infty} f(z)\,\delta z = 1 \tag{52}$$

B. Using Equations (8) and (52) above the equation for the expected value of random asset value at time t divided by known asset value at time zero is...

$$\mathbb{E}\left[\frac{A_i(t)}{A_i(0)}\right] = \int\limits_{-\infty}^{\infty} f(Z_i) \int\limits_{-\infty}^{\infty} f(Z_c)\,\text{Exp}\left\{\lambda_i\,t + \rho_i\,\sigma_i\sqrt{t}\,Z_c + \sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_c\,\delta Z_i \tag{53}$$

We will define the variable I to be the inner integral of Equation (53) above. The solution to that integral is...

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{\lambda_i\,t+\rho_i\,\sigma_i\sqrt{t}\,Z_c+\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_c \\
&= \int_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{\lambda_i\,t+\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_c \\
&= \mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\int_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{\lambda_i\,t+\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\delta Z_c
\end{aligned}
\tag{54}
$$

The solution to the integral in Equation (54) above is...

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{\lambda_i\,t+\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\delta Z_c &= \int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z_c^2\right\}\mathrm{Exp}\left\{\lambda_i\,t+\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\delta Z_c \\
&= \int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z_c^2+\lambda_i\,t+\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\delta Z_c \\
&= \mathrm{Exp}\left\{\lambda_i\,t\right\}\int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\left(Z_c^2-2\,\rho_i\,\sigma_i\sqrt{t}\,Z_c\right)\right\}\delta Z_c
\end{aligned}
\tag{55}
$$

We will define the variable θ as follows...

$$
\theta = Z_c-\rho_i\,\sigma_i\sqrt{t}\ \text{ ...and... }\ \theta^2 = Z_c^2-2\,\rho_i\,\sigma_i\sqrt{t}\,Z_c+\rho_i^2\sigma_i^2\,t
\tag{56}
$$

Using Equation (56) above we can rewrite Equation (55) above as...

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{\lambda_i\,t+\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\delta Z_c &= \mathrm{Exp}\left\{\lambda_i\,t\right\}\int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\theta^2\right\}\mathrm{Exp}\left\{\frac{1}{2}\,\rho_i^2\sigma_i^2\,t\right\}\delta Z_c \\
&= \mathrm{Exp}\left\{\lambda_i\,t+\frac{1}{2}\,\rho_i^2\sigma_i^2\,t\right\}\int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\theta^2\right\}\delta Z_c
\end{aligned}
\tag{57}
$$

Note that the derivative of θ (as defined by Equation (56) above) with respect to the random variable is...

$$
\frac{\delta\theta}{\delta Z_c} = 1\ \text{ ...so... }\ \delta\theta = \delta Z_c
\tag{58}
$$

Using Equations (52) and (58) above we can rewrite Equation (57) above as...

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{\lambda_i\,t+\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\delta Z_c &= \mathrm{Exp}\left\{\lambda_i\,t+\frac{1}{2}\,\rho_i^2\sigma_i^2\,t\right\}\int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\theta^2\right\}\delta\theta \\
&= \mathrm{Exp}\left\{\lambda_i\,t+\frac{1}{2}\,\rho_i^2\sigma_i^2\,t\right\}\ \text{ ...because... }\ \int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\theta^2\right\}\delta\theta = 1
\end{aligned}
\tag{59}
$$

Using Equation (59) above we can rewrite Equation (54) above

$$
I = \mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\mathrm{Exp}\left\{\lambda_i\,t+\frac{1}{2}\,\rho_i^2\sigma_i^2\,t\right\}
\tag{60}
$$

Using Equation (60) above we can rewrite Equation (53) above as...

$$\mathbb{E}\left[\frac{A_i(t)}{A_i(0)}\right] = \int\limits_{-\infty}^{\infty} f(Z_i)\,\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\mathrm{Exp}\left\{\lambda_i\,t+\frac{1}{2}\,\rho_i^2\sigma_i^2\,t\right\}\delta Z_i$$

$$= \mathrm{Exp}\left\{\lambda_i\,t+\frac{1}{2}\,\rho_i^2\sigma_i^2\,t\right\}\int\limits_{-\infty}^{\infty} f(Z_i)\,\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i \tag{61}$$

Using Equation (52) above the solution to the integral in Equation (61) above is...

$$\int\limits_{-\infty}^{\infty} f(Z_i)\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i = \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z_i^2\right\}\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i$$

$$= \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z_i^2+\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i$$

$$= \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\Big(Z_i^2-2\,\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\Big)\right\}\delta Z_i \tag{62}$$

We will redefine the variable θ as follows...

$$\theta = Z_i-\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\ \text{ ...and... }\ \theta^2 = Z_i^2-2\,\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i+(1-\rho_i^2)\,\sigma_i^2\,t \tag{63}$$

Using Equation (63) above we can rewrite Equation (62) above as...

$$\int\limits_{-\infty}^{\infty} f(Z_i)\,\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i = \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\mathrm{Exp}\left\{\frac{1}{2}\,(1-\rho_i^2)\,\sigma_i^2\,t\right\}\delta Z_i$$

$$= \mathrm{Exp}\left\{\frac{1}{2}\,(1-\rho_i^2)\,\sigma_i^2\,t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta Z_i \tag{64}$$

Note that the derivative of θ (as defined by Equation (63) above) with respect to the random variable is...

$$\frac{\delta\theta}{\delta Z_c} = 1\ \text{ ...so... }\ \delta\theta = \delta Z_c \tag{65}$$

Using Equations (52) and (65) above we can rewrite Equation (64) above as...

$$\int\limits_{-\infty}^{\infty} f(Z_i)\,\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i = \mathrm{Exp}\left\{\frac{1}{2}\,(1-\rho_i^2)\,\sigma_i^2\,t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta\theta$$

$$= \mathrm{Exp}\left\{\frac{1}{2}\,(1-\rho_i^2)\,\sigma_i^2\,t\right\}\ \text{ ...because... }\ \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta\theta = 1 \tag{66}$$

Using Equation (66) above we can rewrite Equation (61) above as...

$$\mathbb{E}\left[\frac{A_i(t)}{A_i(0)}\right] = \mathrm{Exp}\left\{\lambda_i\,t+\frac{1}{2}\,\rho_i^2\sigma_i^2\,t\right\}\mathrm{Exp}\left\{\frac{1}{2}\,(1-\rho_i^2)\,\sigma_i^2\,t\right\}$$

$$= \mathrm{Exp}\left\{\lambda_i\,t+\frac{1}{2}\,\rho_i^2\sigma_i^2\,t+\frac{1}{2}\,(1-\rho_i^2)\,\sigma_i^2\,t\right\}$$

$$= \mathrm{Exp}\left\{\lambda_i\,t+\frac{1}{2}\,\sigma_i^2\,t\right\} \tag{67}$$

C. Using Equations (10) and (52) above the equation for expected asset value squared at time t is...

$$\mathbb{E}\left[\frac{A_i(t)^2}{A_i(0)^2}\right] = \int\limits_{-\infty}^{\infty} f(Z_i) \int\limits_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i\,\sigma_i\sqrt{t}\,Z_c + 2\,\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_c\,\delta Z_i \tag{68}$$

We will define the variable I to be the inner integral of Equation (68) above. The solution to that integral is...

$$\begin{aligned} I &= \int\limits_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i\,\sigma_i\sqrt{t}\,Z_c + 2\,\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_c \\ &= \int\limits_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\mathrm{Exp}\left\{2\,\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_c \\ &= \mathrm{Exp}\left\{2\,\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\int\limits_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\delta Z_c \end{aligned} \tag{69}$$

The solution to the integral in Equation (69) above is...

$$\begin{aligned} \int\limits_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\delta Z_c &= \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z_c^2\right\}\mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\delta Z_c \\ &= \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z_c^2 + 2\,\lambda_i\,t + 2\,\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\delta Z_c \\ &= \mathrm{Exp}\left\{2\,\lambda_i\,t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\Big(Z_c^2 - 4\,\rho_i\,\sigma_i\sqrt{t}\,Z_c\Big)\right\}\delta Z_c \end{aligned} \tag{70}$$

We will define the variable θ as follows...

$$\theta = Z_c - 2\,\rho_i\,\sigma_i\sqrt{t}\ \text{ ...and... }\ \theta^2 = Z_c^2 - 4\,\rho_i\,\sigma_i\sqrt{t}\,Z_c + 4\,\rho_i^2\sigma_i^2\,t \tag{71}$$

Using Equation (71) above we can rewrite Equation (70) above as...

$$\begin{aligned} \int\limits_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\delta Z_c &= \mathrm{Exp}\left\{2\,\lambda_i\,t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\theta^2\right\}\mathrm{Exp}\left\{2\,\rho_i^2\sigma_i^2\,t\right\}\delta Z_c \\ &= \mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i^2\sigma_i^2\,t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\theta^2\right\}\delta Z_c \end{aligned} \tag{72}$$

Note that the derivative of θ (as defined by Equation (71) above) with respect to the random variable is...

$$\frac{\delta\theta}{\delta Z_c} = 1\ \text{ ...so... }\ \delta\theta = \delta Z_c \tag{73}$$

Using Equations (52) and (73) above we can rewrite Equation (72) above as...

$$\begin{aligned} \int\limits_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i\,\sigma_i\sqrt{t}\,Z_c\right\}\delta Z_c &= \mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i^2\sigma_i^2\,t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\theta^2\right\}\delta\theta \\ &= \mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i^2\sigma_i^2\,t\right\}\ \text{ ...because... }\ \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\theta^2\right\}\delta\theta = 1 \end{aligned} \tag{74}$$

Using Equation (74) above we can rewrite Equation (69) above

$$I = \mathrm{Exp}\left\{2\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i^2\sigma_i^2\,t\right\} \tag{75}$$

Using Equation (75) above we can rewrite Equation (53) above as...

$$\begin{aligned}\mathbb{E}\left[\frac{A_i(t)^2}{A_i(0)^2}\right] &= \int_{-\infty}^{\infty} f(Z_i)\,\mathrm{Exp}\left\{2\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i^2\sigma_i^2\,t\right\}\delta Z_i \\ &= \mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i^2\sigma_i^2\,t\right\}\int_{-\infty}^{\infty} f(Z_i)\,\mathrm{Exp}\left\{2\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i \end{aligned} \tag{76}$$

Using Equation (52) above the solution to the integral in Equation (76) above is...

$$\begin{aligned}\int_{-\infty}^{\infty} f(Z_i)\mathrm{Exp}\left\{2\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i &= \int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z_i^2\right\}\mathrm{Exp}\left\{2\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i \\ &= \int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z_i^2 + 2\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i \\ &= \int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\left(Z_i^2 - 4\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right)\right\}\delta Z_i \end{aligned} \tag{77}$$

We will redefine the variable θ as follows...

$$\theta = Z_i - 2\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\ \text{ ...and... }\ \theta^2 = Z_i^2 - 4\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i + 4\,(1-\rho_i^2)\,\sigma_i^2\,t \tag{78}$$

Using Equation (78) above we can rewrite Equation (77) above as...

$$\begin{aligned}\int_{-\infty}^{\infty} f(Z_i)\,\mathrm{Exp}\left\{2\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i &= \int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\mathrm{Exp}\left\{2\,(1-\rho_i^2)\,\sigma_i^2\,t\right\}\delta Z_i \\ &= \mathrm{Exp}\left\{2\,(1-\rho_i^2)\,\sigma_i^2\,t\right\}\int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta Z_i \end{aligned} \tag{79}$$

Note that the derivative of θ (as defined by Equation (78) above) with respect to the random variable is...

$$\frac{\delta\theta}{\delta Z_c} = 1\ \text{ ...so... }\ \delta\theta = \delta Z_c \tag{80}$$

Using Equations (52) and (80) above we can rewrite Equation (79) above as...

$$\begin{aligned}\int_{-\infty}^{\infty} f(Z_i)\,\mathrm{Exp}\left\{2\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i &= \mathrm{Exp}\left\{2\,(1-\rho_i^2)\,\sigma_i^2\,t\right\}\int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta\theta \\ &= \mathrm{Exp}\left\{2\,(1-\rho_i^2)\,\sigma_i^2\,t\right\}\ \text{ ...because... }\ \int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta\theta = 1 \end{aligned} \tag{81}$$

Using Equation (81) above we can rewrite Equation (76) above as...

$$\begin{aligned}\mathbb{E}\left[\frac{A_i(t)^2}{A_i(0)^2}\right] &= \mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i^2\sigma_i^2\,t\right\}\mathrm{Exp}\left\{2\,(1-\rho_i^2)\,\sigma_i^2\,t\right\} \\ &= \mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\rho_i^2\sigma_i^2\,t + 2\,(1-\rho_i^2)\,\sigma_i^2\,t\right\} \\ &= \mathrm{Exp}\left\{2\,\lambda_i\,t + 2\,\sigma_i^2\,t\right\} \end{aligned} \tag{82}$$

D. Using Equations (13) and (52) above the equation for the expected value of the product of two asset prices at time t is...

$$\mathbb{E}\left[\frac{A_i(t)\,A_j(t)}{A_i(0)\,A_j(0)}\right] = \int\limits_{-\infty}^{\infty} f(Z_j) \int\limits_{-\infty}^{\infty} f(Z_i) \int\limits_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)Z_c + \sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i + \sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\right\}\delta Z_c\,\delta Z_i\,\delta Z_j \tag{83}$$

We will define the variable I to be the inner integral in Equation (83) above. The solution to that integral is...

$$\begin{aligned} I &= \int\limits_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)Z_c + \sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i + \sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\right\}\delta Z_c \\ &= \int\limits_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)Z_c\right\}\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i + \sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\right\}\delta Z_c \\ &= \mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i + \sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\right\}\int\limits_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)Z_c\right\}\delta Z_c \end{aligned} \tag{84}$$

We will define the variable I_x to be the integral in Equation (84) above. The equation for that integral is...

$$I_x = \int\limits_{-\infty}^{\infty} f(Z_c)\,\mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)Z_c\right\}\delta Z_c \tag{85}$$

Using Equation (52) above the solution to Equation (85) above is...

$$\begin{aligned} I_x &= \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z_c^2\right\}\mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)Z_c\right\}\delta Z_c \\ &= \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z_c^2 + \left(\lambda_i + \lambda_j\right)t + \left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)Z_c\right\}\delta Z_c \\ &= \mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t\right\}\int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z_c^2 + \left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)Z_c\right\}\delta Z_c \end{aligned} \tag{86}$$

We will define the variable θ as follows...

$$\theta = Z_c - \left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right) \text{ ...and... } \theta^2 = Z_c^2 - 2\left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)Z_c + \left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)^2 \tag{87}$$

Using Equation (87) above we can rewrite Equation (86) above as...

$$\begin{aligned} I_x &= \mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t\right\}\int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\theta^2\right\}\mathrm{Exp}\left\{\frac{1}{2}\left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)^2\right\}\delta Z_c \\ &= \mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \frac{1}{2}\left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)^2\right\}\int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\theta^2\right\}\delta Z_c \end{aligned} \tag{88}$$

Note that the derivative of θ (as defined by Equation (87) above) with respect to the random variable is...

$$\frac{\delta\theta}{\delta Z_c} = 1 \text{ ...so... } \delta\theta = \delta Z_c \tag{89}$$

Using Equations (52) and (89) above we can rewrite Equation (88) above as...

$$
\begin{aligned}
I_x &= \mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \frac{1}{2}\left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)^2\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\theta^2\right\}\delta\theta \\
&= \mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \frac{1}{2}\left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)^2\right\}\ \text{...because...}\ \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\theta^2\right\}\delta\theta = 1
\end{aligned} \tag{90}
$$

Using Equation (90) above we can rewrite Equation (84) above

$$
\begin{aligned}
I &= A_i(0)\,A_j(0)\,\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i + \sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\right\}I_x \\
&= A_i(0)\,A_j(0)\,\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i + \sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\right\}\mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \frac{1}{2}\left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)^2\right\}
\end{aligned} \tag{91}
$$

Using Equation (91) above we can rewrite Equation (83) above as...

$$
\begin{aligned}
\mathbb{E}\left[\frac{A_i(t)\,A_j(t)}{A_i(0)\,A_j(0)}\right] &= \int\limits_{-\infty}^{\infty} f(Z_j)\int\limits_{-\infty}^{\infty} f(Z_i)\,\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i + \sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\right\} \\
&\quad \mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \frac{1}{2}\left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)^2\right\}\delta Z_i\,\delta Z_j \\
&= \int\limits_{-\infty}^{\infty} f(Z_j)\,\mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \frac{1}{2}\left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)^2\right\} \\
&\quad \int\limits_{-\infty}^{\infty} f(Z_i)\,\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\mathrm{Exp}\left\{\sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\right\}\delta Z_i\,\delta Z_j \\
&= \int\limits_{-\infty}^{\infty} f(Z_j)\,\mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \frac{1}{2}\left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)^2\right\}\mathrm{Exp}\left\{\sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\right\} \\
&\quad \int\limits_{-\infty}^{\infty} f(Z_i)\,\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i\,\delta Z_j
\end{aligned} \tag{92}
$$

Using Equation (66) above the solution to the inner integral in Equation (92) above is...

$$
\int\limits_{-\infty}^{\infty} f(Z_i)\,\mathrm{Exp}\left\{\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\right\}\delta Z_i = \mathrm{Exp}\left\{\frac{1}{2}\,(1-\rho_i^2)\,\sigma_i^2\,t\right\} \tag{93}
$$

Using Equation (93) above we can rewrite Equation (92) above as...

$$
\begin{aligned}
\mathbb{E}\left[\frac{A_i(t)\,A_j(t)}{A_i(0)\,A_j(0)}\right] &= \int\limits_{-\infty}^{\infty} f(Z_j)\,\mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \frac{1}{2}\left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)^2\right\} \\
&\quad \mathrm{Exp}\left\{\frac{1}{2}\,(1-\rho_i^2)\,\sigma_i^2\,t\right\}\mathrm{Exp}\left\{\sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\right\}\delta Z_j \\
&= \mathrm{Exp}\left\{\left(\lambda_i + \lambda_j\right)t + \frac{1}{2}\left(\rho_i\,\sigma_i\sqrt{t} + \rho_j\,\sigma_j\sqrt{t}\right)^2\right\}\mathrm{Exp}\left\{\frac{1}{2}\,(1-\rho_i^2)\,\sigma_i^2\,t\right\} \\
&\quad \int\limits_{-\infty}^{\infty} f(Z_j)\,\mathrm{Exp}\left\{\sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\right\}\delta Z_j
\end{aligned} \tag{94}
$$

Using Equation (66) above the solution to the integral in Equation (94) above is...

$$\int\limits_{-\infty}^{\infty} f(Z_j)\,\mathrm{Exp}\left\{\sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\right\}\delta Z_j = \mathrm{Exp}\left\{\frac{1}{2}\,(1-\rho_j^2)\,\sigma_j^2\,t\right\} \tag{95}$$

Using Equation (95) above we can rewrite Equation (94) above as...

$$\mathbb{E}\left[\frac{A_i(t)A_j(t)}{A_i(0)A_j(0)}\right] = \mathrm{Exp}\left\{\left(\lambda_i+\lambda_j\right)t+\frac{1}{2}\left(\rho_i\,\sigma_i\sqrt{t}+\rho_j\,\sigma_j\sqrt{t}\right)^2\right\}\mathrm{Exp}\left\{\frac{1}{2}\,(1-\rho_i^2)\,\sigma_i^2\,t\right\}\mathrm{Exp}\left\{\frac{1}{2}\,(1-\rho_j^2)\,\sigma_j^2\,t\right\} \tag{96}$$

E. Using Equation (17) above the equation for the first moment of the distribution of the i'th asset's random rate of return is...

$$\begin{aligned}\mathbb{E}\Big[r_i(t)\Big] &= \mathbb{E}\Big[\lambda_i\,t+\rho_i\,\sigma_i\sqrt{t}\,Z_c+\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\Big]\\ &= \lambda_i\,t+\rho_i\,\sigma_i\sqrt{t}\,\mathbb{E}\Big[Z_c\Big]+\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,\mathbb{E}\Big[Z_i\Big]\\ &= \lambda_i\,t\ \text{ ...because... }\ \mathbb{E}\Big[Z_c\Big]=\mathbb{E}\Big[Z_i\Big]=0\end{aligned} \tag{97}$$

F. Using Equation (17) above the equation for the second moment of the distribution of the i'th asset's random rate of return is...

$$\begin{aligned}\mathbb{E}\Big[r_i(t)^2\Big] &= \mathbb{E}\Big[\Big(\lambda_i\,t+\rho_i\,\sigma_i\sqrt{t}\,Z_c+\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\Big)^2\Big]\\ &= \mathbb{E}\Big[\lambda_i^2\,t^2+\rho_i^2\,\sigma_i^2\,t\,Z_c^2+(1-\rho_i^2)\,\sigma_i^2\,t\,Z_i^2\Big]\\ &= \lambda_i^2\,t^2+\rho_i^2\,\sigma_i^2\,t\,\mathbb{E}\Big[Z_c^2\Big]+(1-\rho_i^2)\,\sigma_i^2\,t\,\mathbb{E}\Big[Z_i^2\Big]\\ &= \lambda_i^2\,t^2+\sigma_i^2\,t\ \text{ ...because... }\ \mathbb{E}\Big[Z_c^2\Big]=\mathbb{E}\Big[Z_i^2\Big]=1\end{aligned} \tag{98}$$

Note: Given that $\mathbb{E}[Z_c]=\mathbb{E}[Z_i]=\mathbb{E}[Z_c\,Z_i]=0$ we can ignore all variables that include Z_c, Z_i, and the product of Z_c and Z_i.

G. Using Equation (17) above the expected product of the rates of return on the i'th and j'th asset in the portfolio over the time interval $[0,t]$ is...

$$\begin{aligned}\mathbb{E}\Big[r_i(t)\,r_j(t)\Big] &= \mathbb{E}\Big[\Big(\lambda_i\,t+\rho_i\,\sigma_i\sqrt{t}\,Z_c+\sqrt{1-\rho_i^2}\,\sigma_i\sqrt{t}\,Z_i\Big)\Big(\lambda_j\,t+\rho_j\,\sigma_j\sqrt{t}\,Z_c+\sqrt{1-\rho_j^2}\,\sigma_j\sqrt{t}\,Z_j\Big)\Big]\\ &= \mathbb{E}\Big[\lambda_i\,\lambda_j\,t^2+\rho_i\,\rho_j\sigma_i\,\sigma_j\,t\,Z_c^2\Big]\\ &= \lambda_i\,\lambda_j\,t^2+\rho_i\,\rho_j\sigma_i\,\sigma_j\,t\,\mathbb{E}\Big[Z_c^2\Big]\\ &= \lambda_i\,\lambda_j\,t^2+\rho_i\,\rho_j\,\sigma_i\,\sigma_j\,t\ \text{ ...because... }\ \mathbb{E}\Big[Z_c^2\Big]=1\end{aligned} \tag{99}$$

Note: Given that $\mathbb{E}[Z_c\,Z_i]=\mathbb{E}[Z_c\,Z_j]=\mathbb{E}[Z_i\,Z_j]=0$ we can ignore all variables that include these products.